BER evaluation for general QAM in Nakagami-*m* fading channels

G. Aniba and S. Aïssa

Evaluation of the bit error rate for general M-ary quadrature amplitude modulation (M-QAM) in Nakagami-m fading channels is presented. The analysis considers real values of the Nakagami fading parameter m, and bit-to-symbol mapping that is not necessarily Gray. Analytical and simulation results are compared to illustrate the accuracy of the analysis, taking as examples non-Gray mapped 16-QAM and 32-cross-QAM constellations.

Introduction: Bit error rate (BER) represents a reference metric widely used in the performance evaluation of M-ary quadrature amplitude modulation (M-QAM). In this context, BER analysis generally considers square M-QAM constellations and Gray coding for the mapping of bits into symbols. Indeed, considering general M-QAM, few studies presented exact closed-form expressions for the BER, and only environments with AWGN, Rayleigh fading, or Nakagami-m fading with integer m were therein considered. For instance, a BER analysis in AWGN channel considering M-QAM with arbitrary mapping was provided in [1], and a symbol error probability (SEP) analysis of rectangular M-QAM assuming Gray coding and Rayleigh fading was presented in [2]. Considering Nakagami-m fading, on the other hand, the SEP of general order rectangular QAM was recently evaluated in [3] for integer values of the Nakagami parameter m. Recently, analysis applied to rectangular QAM considering $m = k + 1/2$, where k takes integer values, was presented in [4, 5]. In this Letter, we present analytical formulation for the BER considering general M-QAM, square or rectangular and not necessarily Gray coded, over Nakagami-m fading channels with arbitrary real parameter $m \geq 1/2$. The analysis is also applied to the 32-cross-QAM case. The latter modulation was recently considered in [6] where exact expressions for the SEP in slow Nakagami-m fading channels were presented. However, only integer values of m were considered in [6]. In this Letter, we extend our analysis for general QAM to the case with 32-cross-QAM and evaluate the BER when the Nakagami fading parameter is not necessarily an integer.

BER evaluation: Consider transmission of an M-QAM signal over a Nakagami-m fading channel. The transmitted symbols are denoted by $s = s_r + js_i$ and the decoded ones by $\tilde{s} = \tilde{s}_r + j\tilde{s}_i$. Let d_r and d_i denote the minimum distances in the real (r) and imaginary (i) dimensions, respectively, and $d_H(s, \tilde{s})$ be the Hamming distance between the codewords corresponding to symbols s and $\tilde{s}$. The average BER is given by

$$\mathrm{BER} = \frac{1}{M\log_2 M}\sum_{s\in\mathcal{S}}\sum_{\tilde{s}\in\mathcal{S},\tilde{s}\neq s} d_H(s,\tilde{s}) \times \int_0^\infty f(m,\gamma)g(s_r,d_r,\gamma)g(s_i,d_i,\gamma)d\gamma \tag{1}$$

with

$$g(s,d,\gamma) = Q\Big((|s-\tilde{s}| - d/2)\sqrt{2\gamma}\Big) - \delta_{\tilde{s}}Q \times \Big((|s-\tilde{s}| + d/2)\sqrt{2\gamma}\Big) \tag{2}$$

where $\mathcal{S}$ denotes the constellation, $|\cdot|$ indicates the absolute value operator, $\delta_{\tilde{s}}$ is equal to one if the symbol is located on the border of the constellation and equals zero otherwise, the Gaussian $Q(\cdot)$ function is defined by $Q(t) = 1/\sqrt{2\pi}\int_t^\infty e^{-u^2/2}du$, and $f(m,\gamma) = (1/\Gamma(m))(m/\Omega)^m\gamma^{m-1}e^{-m\gamma/\Omega}$ is the PDF of the SNR γ for the Nakagami channel with fading parameter $m \geq 1/2$, corresponding to a gamma distribution, where $\Gamma(\cdot)$ is the gamma function and Ω is the average fading power. As observed, the BER expression (1) involves integral terms of the form

$$I(\alpha_1,\alpha_2) = \int_0^\infty f(m,\gamma)Q(\alpha_1\sqrt{\gamma})Q(\alpha_2\sqrt{\gamma})d\gamma \tag{3}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. To the best of the authors' knowledge, there is no closed-form representation of (3) that accounts for real values of m [2, 3, 6, 7]. In this Letter, we present a closed-form solution for this integral considering real-valued m. This is a useful result, which can also be used in other contexts of BER and SEP analysis, such as in schemes with iterative demodulation methods [8] and cross-QAM constellations.

Theorem 1: A closed-form solution for the integral $I(\alpha_1, \alpha_2)$ considering that m can take arbitrary real values not less than $1/2$ is given by

$$\begin{aligned} I(\alpha_1,\alpha_2) = \frac{1}{4}\Bigg[& 1 - \sqrt{\frac{2}{\pi}}\sqrt{\frac{\alpha_1^2\Omega}{m}}\frac{\Gamma(m+(1/2))}{\Gamma(m)(1+(\Omega\alpha_1^2/2m)^{m+(1/2)}} \\ & \times {}_2F_1\left(1, m+\frac{1}{2}; \frac{3}{2}; \frac{\alpha_1^2}{(2m/\Omega+\alpha_1^2)}\right) \\ & - \sqrt{\frac{2}{\pi}}\sqrt{\frac{\alpha_2^2\Omega}{m}}\frac{\Gamma(m+(1/2))}{\Gamma(m)(1+(\Omega\alpha_2^2/2m)^{m+(1/2)}} \\ & \times {}_2F_1\left(1, m+\frac{1}{2}; \frac{3}{2}; \frac{\alpha_2^2}{((2m/\Omega)+\alpha_2^2)}\right) \\ & + \frac{2\alpha_1\alpha_2\Omega}{\pi(1+(\Omega/2m)(\alpha_1^2+\alpha_2^2))^{m+1}}F_2 \\ & \times \left(m+1, 1, 1; \frac{3}{2}, \frac{3}{2}; \frac{\alpha_1^2}{((2m/\Omega)+\alpha_1^2+\alpha_2^2)}, \frac{\alpha_2^2}{((2m/\Omega)+\alpha_1^2+\alpha_2^2)}\right)\Bigg] \end{aligned} \tag{4}$$

where ${}_2F_1(a, b; c; x)$ is a Gauss hypergeometric function and $F_2(a, b, b'; c, c'; x, y)$ is an Appell hypergeometric function [9].

Proof: First, express the Gaussian function as $Q(t) = 1/2(1-\mathrm{erf}(t/\sqrt{2}))$ where $\mathrm{erf}(t) = (2/\sqrt{\pi}\int_0^t e^{-u^2}du$, we have $Q(\alpha_1\sqrt{\gamma})Q(\alpha_1\sqrt{\gamma}) = 1/4(1-\mathrm{erf}\ (\alpha_1\sqrt{\gamma/2}))(1-\mathrm{erf}(\alpha_2\sqrt{\gamma/2}))$.
Thus, (3) can be rewritten as

$$I(\alpha_1,\alpha_2) = \frac{1}{4}\frac{1}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m[f_1 - f_2(\alpha_1) - f_2(\alpha_2) + f_3(\alpha_1,\alpha_2)] \tag{5}$$

where

$$f_1 = \int_0^\infty \gamma^{m-1}e^{-(m/\Omega)\gamma}d\gamma = \left(\frac{\Omega}{m}\right)^m\Gamma(m) \tag{6}$$

$$f_2(\alpha) = \int_0^\infty \gamma^{m-1}\mathrm{erf}(\alpha\sqrt{\gamma/2})e^{-(m/\Omega)\gamma}d\gamma \tag{7}$$

$$f_3(\alpha_1,\alpha_2) = \int_0^\infty \gamma^{m-1}\mathrm{erf}(\alpha_1\sqrt{\gamma/2}) \times \mathrm{erf}(\alpha_2\sqrt{\gamma/2})e^{-(m/\Omega)\gamma}d\gamma \tag{8}$$

The function $\mathrm{erf}(\cdot)$ can further be written in terms of a hypergeometric function as [10, eqn. (7.1.21)]:

$$\mathrm{erf}(t) = \frac{2t}{\sqrt{\pi}}e^{-t^2}{}_1F_1\left(1; \frac{3}{2}; t^2\right) \tag{9}$$

where ${}_1F_1(a; b; t)$ is also known as Kummer's function of the first kind, and is defined as the series ${}_1F_1(a; b; t) \triangleq \Sigma_{k=0}^\infty((a)_k/(b)_k)(t^k/k!)$. Hence, substituting (9) into (7) yields

$$f_2(\alpha) = \alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \gamma^{m-(1/2)}{}_1F_1\left(1; \frac{3}{2}; \frac{\alpha^2}{2}\gamma\right)e^{-(m/\Omega+\alpha^2/2)\gamma}d\gamma \tag{10}$$

which, making use of the unilateral Laplace transform defined by $\mathcal{L}[g(t)](p) = \int_0^\infty g(t)e^{-pt}dt$, can be expressed as

$$f_2(\alpha) = \alpha\sqrt{\frac{2}{\pi}}\mathcal{L}\left[\gamma^{m-(1/2)}{}_1F_1\left(1; \frac{3}{2}; \frac{\alpha^2}{2}\gamma\right)\right]\left(\frac{m}{\Omega}+\frac{\alpha^2}{2}\right) \tag{11}$$

Then, based on the Laplace table [11, eqn. (3.35.1.2)], we have $\mathcal{L}[\gamma^\mu{}_1F_1(a; b; \omega\gamma)](p) = (\Gamma(\mu+1)/p^{\mu+1}){}_2F_1(a, \mu+1; b; (\omega/p))$ where $\mathcal{R}e[\mu] > -1$; $\mathcal{R}e[p]$, $\mathcal{R}e[p-\omega] > 0$. Accordingly, (11) can be reformulated as

$$f_2(\alpha) = \alpha\sqrt{\frac{2}{\pi}}\frac{\Gamma(m+(1/2))}{(m/\Omega+\alpha^2/2)^{m+1/2}}\,{}_2F_1 \times \left(1, m+\frac{1}{2}; \frac{3}{2}; \frac{\alpha^2}{((2m/\Omega)+\alpha^2)}\right) \tag{12}$$

where ${}_2F_1(a, b; c; t) \triangleq \sum_{k=0}^{\infty}((a)_k(b)_k/(c)_k)(t^k/k!)$ [9] with $(\cdot)_k$ denoting the Pochhammer symbol given by $(x)_k \triangleq (\Gamma(x+k)/\Gamma(x)) = x(x+1)\ldots(x+k-1)$.

Now in order to compute $f_3(\alpha_1, \alpha_2)$, we follow the same procedure as the one used for $f_2(\alpha)$. Indeed, by using (9) in (8), the latter can be expressed as

$$\begin{aligned} f_3(\alpha_1, \alpha_2) &= \alpha_1\alpha_2\frac{2}{\pi}\int_0^{\infty}\gamma^m\,{}_1F_1\left(1; \frac{3}{2}; \frac{\alpha_1^2}{2}\gamma\right){}_1F_1 \\ &\quad \times \left(1; \frac{3}{2}; \frac{\alpha_2^2}{2}\gamma\right)e^{-((m/\Omega)+(\alpha_1^2/2)+(\alpha_2^2/2))\gamma}d\gamma \\ &= \alpha_1\alpha_2\frac{2}{\pi}\mathcal{L}\left[\gamma^m\,{}_1F_1\left(1; \frac{3}{2}; \frac{\alpha_1^2}{2}\gamma\right){}_1F_1\right. \\ &\quad \left.\times \left(1; \frac{3}{2}; \frac{\alpha_2^2}{2}\gamma\right)\right]\left(\frac{m}{\Omega}+\frac{\alpha_1^2}{2}+\frac{\alpha_2^2}{2}\right) \end{aligned} \tag{13}$$

In this case, we use the following Laplace transform [11, eqn. (3.35.7.1)]:

$$\mathcal{L}[\gamma_1^{\mu}F_1(a'; b'; \sigma\gamma)_1F_1(a; b; \omega\gamma)](p) = \frac{\Gamma(\mu+1)}{p^{\mu+1}}F_2 \times \left(\mu+1, a, a'; b, b'; \frac{\omega}{p}, \frac{\sigma}{p}\right)$$

where $F_2(a, b, b'; c, c'; x, y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}((a)_{k+n}(b)_k(b')_n/k!n!(c)_k(c')_n)\,x^ky^n$ and $\mathcal{R}e[\mu] > -1$; $\mathcal{R}e[p]$, $\mathcal{R}e[p-\sigma]$, $\mathcal{R}e[p-\omega]$, $\mathcal{R}e[p-\sigma-\omega] > 0$ to express (13) as

$$\begin{aligned} f_3(\alpha_1, \alpha_2) &= \frac{2\alpha_1\alpha_2}{\pi}\frac{\Gamma(m+1)}{((m/\Omega)+(\alpha_1^2/2)+(\alpha_2^2/2))^{m+1}}F_2 \\ &\quad \times \left(m+1, 1, 1; \frac{3}{2}, \frac{3}{2}; \frac{\alpha_1^2}{((2m/\Omega)+\alpha_1^2+\alpha_2^2)},\right. \\ &\quad \left.\times \frac{\alpha_2^2}{((2m/\Omega)+\alpha_1^2+\alpha_2^2)}\right) \end{aligned} \tag{14}$$

Finally, using (5), (6), (12) and (14), the integral $I(\alpha_1, \alpha_2)$ can be written in closed-form as shown in (4).

The Gauss's hypergeometric function ${}_2F_1$ is implemented in standard softwares such as Matlab, Maple and Mathematica. Moreover, the Appell hypergeometric function F_2 can be implemented using $F_2(a, b, b'; c, c'; x, y) = \sum_{k=0}^{\infty}((a)_k(b)_k/k!(c)_k)\,x^k\,{}_2F_1(a+k, b'; c'; y)$ [12], which can further be simplified when $b = b' = 1$ considering that $(b)_k = k!$ and $(b')_n = n!$. Derivation of (1) can then be directly deduced considering (2) and using the closed-form expression for the integral (3) as given by (4). The latter result can also be applied for cross-QAM modulations. In particular, in the 32-cross-QAM case, the SER can be expressed as $P_e = (1/32)[4I_1(1/\sqrt{5}) + 104I_1(1/\sqrt{10}) - 92I_2(1/\sqrt{10})]$ [6] where $I_1(\alpha) = \int_0^{+\infty}Q(\alpha\sqrt{\gamma})f(m, \gamma)d\gamma$ and $I_2(\alpha) = \int_0^{+\infty}Q(\alpha\sqrt{\gamma})^2f(m, \gamma)d\gamma$. The BER of the 32-cross-QAM in Nakagami fading channels with real m can also be formulated using these expressions ($I_1(\alpha)$ and $I_2(\alpha)$) and making use of the same error decision regions as in [6, 7]. Derivation of these integrals can easily be obtained from the above analysis. Indeed, using (6) and (12), $I_1(\alpha)$ can be expressed as

$$I_1(\alpha) = \frac{1}{2}\left[1-\sqrt{\frac{2}{\pi}}\sqrt{\frac{\alpha^2\Omega}{m}}\frac{\Gamma(m+(1/2))}{\Gamma(m)(1+(\Omega\alpha^2/2m))^{m+(1/2)}}\,{}_2F_1 \times \left(1, m+\frac{1}{2}; \frac{3}{2}; \frac{\alpha^2}{((2m/\Omega)+\alpha^2)}\right)\right] \tag{15}$$

and $I_2(\alpha)$, being a special case of (3), can directly be obtained using $I_2(\alpha) = I(\alpha, \alpha)$ in (4).

Numerical results: To show the accuracy of the proposed formulation for the BER, simulations were conducted considering two schemes, square 16-QAM with non-Gray mapping and 32-cross-QAM, and several illustrative values for the Nakagami fading parameter $1 \leq m \leq 10$. Fig. 1*a* and Fig. 1*b* illustrate the accuracy of our analytical results, which match almost perfectly the simulation results for the whole range of simulated SNR.

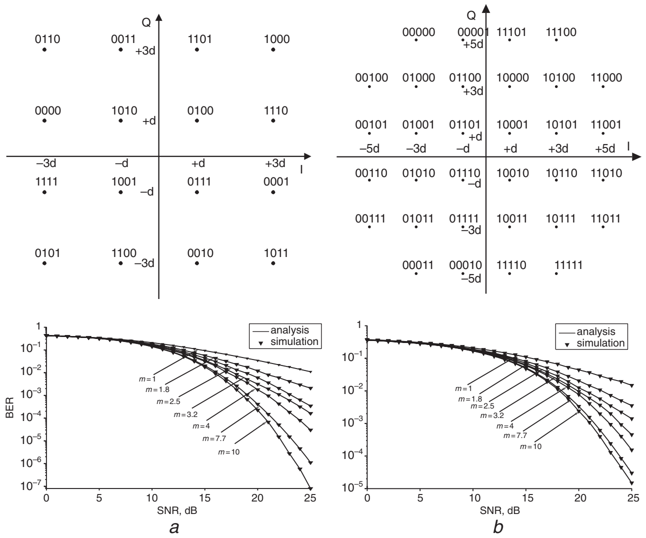

Fig. 1 *Comparison between analytical and simulation results, considering different values for the Nakagami parameter m*
a 16-QAM
b 32-cross-QAM

Conclusions: A BER analysis for general M-QAM considering transmission over Nakagami-m fading channels with real-valued parameter $m \geq 1/2$ is presented. In particular, a closed-form solution for an integral involved in the analysis, which is also useful for other wireless communication applications, is presented. The integral is expressed in terms of the hypergeometric function ${}_2F_1$ and the Appell function F_2, which can easily be evaluated using common numerical software.

19 December 2008
doi: 10.1049/el.2009.3656

G. Aniba and S. Aïssa (*INRS-EMT, University of Quebec, Montreal, QC, Canada*)

E-mail: ghassane@emt.inrs.ca

References

1 Szczecinski, L., Gonzalez, C., and Aïssa, S.: 'Exact expression for the BER of rectangular QAM with arbitrary constellation mapping', *IEEE Trans. Commun.*, 2006, **54**, pp. 389–392
2 Beaulieu, N.C.: 'A useful integral for wireless communication theory and its application to rectangular signaling constellation error rates,', *IEEE Trans. Commun.*, 2006, **54**, pp. 802–805
3 Karagiannidis, G.K.: 'On the symbol error probability of general order rectangular QAM in nakagami-*m* fading', *IEEE Commun. Lett.*, 2006, **10**, pp. 745–747
4 Chen, Y., and Beaulieu, N.C.: 'Solutions to infinite integrals of Gaussian Q-function products and some applications', *IEEE Commun. Lett.*, 2007, **11**, pp. 853–855
5 Hong, X., Mustapha, B., and Leszek, S.: 'Bit error rate for rectangular QAM with arbitrary constellation mapping in nakagami-*m* channels,', *Wirel. Commun. Mob. Comput.*, 2008, **8**, (1), pp. 93–99
6 Beaulieu, N.C., and Chen, Y.: 'Closed-form expressions for the exact symbol error probability of 32-cross-QAM in AWGN and in slow nakagami fading', *IEEE Commun. Lett.*, 2007, **11**, pp. 310–312
7 Vitthaladevuni, P.K., and Alouini, M.S.: 'Exact BER computation for the cross 32-QAM constellation', *Proc. IEEE First Int. Symp. on Control, Communications and Signal Processing (ISCCSP)*, 2004, pp. 643–646
8 Schreckenbach, F., Gortz, N., Hagenauer, J., and Bauch, G.: 'Optimization of symbol mappings for bit-interleaved coded modulation with iterative decoding', *IEEE Commun. Lett.*, 2003, **7**, pp. 593–595

9 Appell, P., and de Friet, J.K.: 'Fonctions hypergomtriques et hypersphriques: polynomes d'Hermite' (Gauthier-Villars, 1926)
10 Abramowitz, M., and Stegun, I.A.: 'Handbook of mathematical functions with formulas, graphs, and mathematical tables' (Dover Publications, New York, NY, USA, 1972, 10th ed.)
11 Prudnikov, A.P., Brychkov, Y.A., and Marichev, O.I.: 'Integrals and series, direct Laplace transforms' (Gordon & Breach Sci. Publ, 1992, vol. 4)
12 'Integrals and Series, More Special Functions' (Gordon & Breach Sci. Publ, 1989, vol. 3)